Proof of the exercise 25.1-4

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Theorem 1. *The matrix multiplication* $(X, Y)_{ij} = \min_k(x_{ik} + y_{kj})$ *defined by EXTEND-SHORTEST-PATHS is associative.*

Proof. Let X, Y, Z be three $n \times n$ matrixs, we shall prove that $\big((X, Y), Z\big) = \big(X, (Y, Z)\big)$.

For a given pair of i and j, since $(X, Y)_{ij} = \min_k(x_{ik} + y_{kj})$, we have

$$\begin{aligned}\big((X,Y),Z\big)_{ij} &= \min_{k'}\Big(\min_k(x_{ik} + y_{kk'}) + z_{k'j}\Big)\\ &= \min_{k'}\min_k(x_{ik} + y_{kk'} + z_{k'j}).\end{aligned}$$

The last equality is due to the fact that $z_{k'j}$ does not depend on k. On the other hand, we have $(Y, Z)_{ij} = \min_k(y_{ik} + z_{kj})$ and

$$\begin{aligned}\big(X,(Y,Z)\big)_{ij} &= \min_{k'}\Big(x_{ik'} + \min_k(y_{k'k} + z_{kj})\Big)\\ &= \min_{k'}\min_k(x_{ik'} + y_{k'k} + z_{kj}).\end{aligned}$$

By an exchange of variables $k \leftrightarrow k'$, we can conclude that $\Big((X,Y),Z\Big)_{ij} = \Big(X,(Y,Z)\Big)_{ij}$. Since the equality is true for any pair of i and j, we have proven that $\big((X,Y),Z\big) = \big(X,(Y,Z)\big)$. Thus the matrix multiplication is associative. □